

# Darboux integrability of determinant and equations for principal minors

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**Abstract.** We consider equations that represent a constancy condition for a 2D Wronskian, mixed Wronskian-Casoratian and 2D Casoratian. These determinantal equations are shown to have the number of independent integrals equal to their order - this implies Darboux integrability. On the other hand, the recurrent formulas for the leading principal minors are equivalent to the 2D Toda equation and its semi-discrete and lattice analogues with particular boundary conditions (cut-off constraints). This connection is used to obtain recurrent formulas and closed-form expressions for integrals of the Toda-type equations from the integrals of the determinantal equations. General solutions of the equations corresponding to vanishing determinants are given explicitly while in the non-vanishing case they are given in terms of solutions of ordinary linear equations.

## 1. Introduction

Consider the scalar equation

$$\det \mathcal{A} = \beta, \quad (1)$$

where  $\beta = \text{const}$  and the entries  $a_{ij}$  of matrix  $\mathcal{A}$  have one of the forms

$$a_{i+1,j+1} = \frac{\partial^{i+j}}{\partial x^i \partial t^j} u(t, x), \quad a_{i+1,j+1} = \frac{d^j}{dt^j} u(t, k+i), \quad a_{i+1,j+1} = u(k+i, l+j)$$

with  $i, j = 0, \dots, N-1$ . The independent variables  $t$  and  $x$  can be either real or complex, while  $k$  and  $l$  are integers. These three forms give rise to correspondingly the partial differential, semi-discrete (differential-difference) and lattice equations. The left-hand side of (1) in the continuous case is sometimes referred to as a 2D Wronskian, since it can be interpreted as either of the Wronskians:  $W(u, u_t, \dots, u_{t \dots t})(x)$  or  $W(u, u_x, \dots, u_{x \dots x})(t)$ . Similarly, in the semi-discrete case it can be regarded as either the Wronskian  $W(u_k, \dots, u_{k+N-1})(t)$  or the Casoratian  $C(u, u^{(1)}, \dots, u^{(N-1)})(k)$ . The lattice case gives rise to an object we will refer to as a 2D Casoratian. These objects naturally arise in construction of solutions of Toda-type equations, both continuous and discrete (see e.g. [1]-[3]).

Apart from scalar equation (1), we are concerned with systems of equations that can be derived from (1). We are particularly interested in the system obtained by denoting  $w_n$  as the leading principal minors of matrix  $\mathcal{A}$ . The first publications on continuous case date back to Darboux [4], who found that  $w_n$  satisfies the chain of equations

$$\partial_{tx}^2 \ln w_n = \frac{w_{n+1}w_{n-1}}{w_n^2}, \quad n = 1 \dots N-1 \quad (2)$$

with the boundary conditions

$$w_0 = 1, \quad (3)$$

$$w_N = \beta, \quad (4)$$

where (4) is simply another representation of equation (1). System (2) is equivalent to a two-dimensional (2D) Toda equation which arguably received more attention than equation (1) itself. Nevertheless equation (1) is, due to its transparent algebraic structure, very convenient for studying certain properties of system (2). For instance, the *local* integrals of (1) have simple expression in terms of minors of matrix  $\mathcal{A}$ . The integrals of system (2) and its reductions can then be derived from the integrals of the scalar equation as demonstrated in [5] for the continuous case. In this paper we elaborate on this result deriving integrals in a similar fashion for the semi-discrete and lattice cases. For completeness of exposition we also give proofs in the continuous case.

Note that presence of integrals entirely depends on boundary conditions (3) and (4). Imposing different boundary conditions, for instance the periodic condition  $w_{n+i} = w_n$ , drastically changes the properties of (2). It is well-known [6] that 2D Toda equations with the periodic condition (for  $i > 1$ ) yield solitonic equations. On the other hand, non-linear equations that possess local integrals form a subclass of linearisable equations. The famous example of such an equation is the Liouville equation

$$v_{tx} = -\beta \exp(2v). \quad (5)$$

Indeed, one can easily verify that if  $z$  solves the d'Alembert equation  $z_{tx} = 0$ , then

$$v = \frac{1}{2} \ln \frac{z_x z_t}{-\beta z^2}$$

is the solution of the Liouville equation. Note that (5) can be written in form (1) if we substitute  $v = -\ln u$ . Sometimes the whole class of equations admitting local integrals is referred to as the Liouville-type equations. The other commonly used name is the *Darboux integrable* equations. There is a vast body of knowledge on these equations, including classification results (see e.g. [7, 8, 9, 10] and references therein). Some progress has been also achieved in the study of systems of equations admitting integrals, especially the exponential systems associated with simple Lie algebras [11, 12, 13, 14, 15].

The study of discrete analogues of the Liouville equation started relatively recently [16]. The main feature of such equations is the presence of local integrals just as it is in the continuous case. It has been shown that these equations also have terminating

sequences of the Laplace invariants [16] and finite-dimensional characteristic algebras [17]. Discrete analogues of the 2D Toda equation corresponding to various Cartan matrices were investigated in [18, 19]. A general construction of integrals for semi-discrete and lattice Toda systems of type  $A$  and  $C$  was proposed in [20]. A different approach based on the consistency properties of a particular case of scalar equation (1), namely the lattice equation with  $\beta = 0$ , was introduced in [21]. The consistency approach (see e.g. [22, 23, 24]) was also applied to study equations associated with hyper-determinants [25].

In this paper equation (1) is studied from the view-point of Darboux integrability. Our main objective is construction of integrals and general solutions of this equation. The result is then transferred to the case of the semi-discrete and lattice analogues of the Toda equation with boundary conditions (3) and (4).

The rest of the paper is organized as follows. Firstly, we consider in detail derivation of integrals for the Liouville equation and its differential-difference analogue. The derivation of integrals in the lattice case is similar to the semi-discrete one. The example of the Liouville equation also illustrates our approach to integrals of equation (1) which is developed in the following sections. The concluding section contains the outline of a procedure for construction of general solutions.

## 2. Integrals of the Liouville equation

The Liouville equation has a remarkable property that it possesses local functions of solutions and their derivatives, let us denote them as  $I(v, v_t, \dots)$  and  $J(v, v_x, \dots)$ , that depend only on one variable along the characteristics:

$$\partial_x I = 0, \quad \partial_t J = 0. \quad (6)$$

These functions are called the  $x$ - and  $t$ -integrals. Note that  $I$  and  $J$  do not depend on  $x$ - and  $t$ -derivatives correspondingly. This is a rather general property (see e.g. [9]) of hyperbolic equations admitting integrals. The simplest integrals of the Liouville equation have the form

$$I = v_{tt} - v_t^2, \quad J = v_{xx} - v_x^2. \quad (7)$$

All other integrals can be shown to be functions of  $I$  and  $J$  and their derivatives. The fulfillment of (6) is verified by differentiating the integrals and eliminating the mixed derivatives by means of (5). In relation to integrals we are also interested in a particular representation of their derivatives

$$\partial_x I = \Lambda_I F, \quad \partial_t J = \Lambda_J F \quad (8)$$

which is reminiscent of characteristic form of conservation laws. Here  $\Lambda_I$  and  $\Lambda_J$  are some differential operators, which we call *integrating factors*, and  $F$  represents the equation in question. For example, the derivatives of (7) can be written as

$$\partial_x I = (\partial_t - 2v_t)(v_{tx} + \beta e^{2v}), \quad \partial_t J = (\partial_x - 2v_x)(v_{tx} + \beta e^{2v}). \quad (9)$$

For lattice equations the analog of (8) is the representation

$$S_k(I) - I = \Lambda_I F, \quad S_l(J) - J = \Lambda_J F, \quad (10)$$

where  $S_k$  and  $S_l$  are the shift operators with respect to the subscripted variables. In the semi-discrete case we have two types of integrals hence the mixture of (8) and (10):

$$S_k(I) - I = \Lambda_I F, \quad \partial_t J = \Lambda_J F. \quad (11)$$

Our interest in representation (8) is motivated by the fact that it contains important information about the structure of co-symmetries and higher symmetries of the equation. For example, for the Liouville equation it can be shown that the formal adjoint operators

$$\Lambda_I^T = -\partial_t - 2v_t, \quad \Lambda_J^T = -\partial_x - 2v_x$$

map correspondingly functions of integrals  $I$  and  $J$  to symmetries of (5). For example, the simplest higher symmetry  $v_\tau = \Lambda_J^T J$  is the mKdV equation in the potential form

$$v_\tau = -v_{xxx} + 2v_x^3.$$

A detailed exposition on construction of symmetries from integrals of hyperbolic equations is discussed in [26]. The representation of the form (10) was used to introduce the notion of *dual equation* in [27]. In this paper we restrict ourselves to finding representations (8), (11) and (10). The problem of construction of higher symmetries from the integrals will be considered elsewhere.

Now we describe how the integrals of the Liouville equation and its discrete counterparts can be obtained from determinantal form (1). As it has been pointed out already, the Liouville equation is equivalent to (1) for  $N = 2$ :

$$\begin{vmatrix} u & u_t \\ u_x & u_{tx} \end{vmatrix} = \beta. \quad (12)$$

Differentiating (12) with respect to  $x$  we obtain

$$\begin{vmatrix} u & u_t \\ u_{xx} & u_{xxt} \end{vmatrix} = 0. \quad (13)$$

Hence the rows of the determinant are proportional:

$$(u_{xx}, u_{xxt}) = J(x)(u, u_t)$$

and the coefficient of proportionality

$$J = \frac{u_{xx}}{u} \quad (14)$$

is a  $t$ -integral of equation (12). Representation (8)<sub>2</sub> is then easily found

$$\partial_t J = \frac{1}{u^2} \partial_x \left( \begin{vmatrix} u & u_t \\ u_x & u_{tx} \end{vmatrix} - \beta \right).$$

Integral (14) takes form (7)<sub>2</sub> on substituting  $u = \exp(-v)$ . The  $t$ -integral is obtained by differentiating (12) with respect to  $t$  and finding the coefficient of proportionality of columns. Obviously, it is the same as (14) with  $x$ - and  $t$ -derivatives interchanged.

*Semi-discrete Liouville equation* We use the determinant form of the Liouville equation to introduce its differential-difference and lattice analogues by simply replacing the derivatives with shifts. Let us consider the differential-difference equation

$$\begin{vmatrix} u_k & \dot{u}_k \\ u_{k+1} & \dot{u}_{k+1} \end{vmatrix} = \beta, \quad (15)$$

where the “dot” indicates the  $t$ -derivative. Naming (15) as a differential-difference Liouville equation is justified because it has (12) as its continuum limit. Indeed, we obtain (12) on setting  $x = k\varepsilon$ ,  $u_k(t) = u(t, x)/\sqrt{\varepsilon}$  in (15) and passing to the limit  $\varepsilon \rightarrow 0$ . Other differential-difference and lattice avatars of the Liouville equation can be found in [16, 28, 29].

On differentiating (15) we obtain

$$\begin{vmatrix} u_k & \ddot{u}_k \\ u_{k+1} & \ddot{u}_{k+1} \end{vmatrix} = 0 \quad (16)$$

which implies the relation

$$(\ddot{u}_k, \ddot{u}_{k+1}) = I(t)(u_k, u_{k+1}).$$

The coefficient of proportionality

$$I = \frac{\ddot{u}_k}{u_k} \quad (17)$$

is the  $k$ -integral of (15). Differencing (17) we find the representation

$$\Delta_k I = \frac{1}{u_k u_{k+1}} \frac{d}{dt} \left( \begin{vmatrix} u_k & \dot{u}_k \\ u_{k+1} & \dot{u}_{k+1} \end{vmatrix} - \beta \right),$$

where

$$\Delta_k = S_k - 1.$$

In order to find the  $t$ -integral of (15) we difference the equation to obtain

$$\begin{vmatrix} u_{k+1} & \dot{u}_{k+1} \\ u_{k+2} & \dot{u}_{k+2} \end{vmatrix} - \begin{vmatrix} u_k & \dot{u}_k \\ u_{k+1} & \dot{u}_{k+1} \end{vmatrix} = 0$$

which can be written as the single determinant

$$\begin{vmatrix} u_{k+1} & \dot{u}_{k+1} \\ u_{k+2} + u_k & \dot{u}_{k+2} + \dot{u}_k \end{vmatrix} = 0. \quad (18)$$

The latter implies

$$(u_{k+2} + u_k, \dot{u}_{k+2} + \dot{u}_k) = J(k)(u_{k+1}, \dot{u}_{k+1})$$

and the following form of the  $t$ -integral:

$$J = \frac{u_{k+2} + u_k}{u_{k+1}}.$$

One can verify that

$$j = \frac{1}{u_{k+1}^2} (S_k - 1) \left( \begin{vmatrix} u_k & \dot{u}_k \\ u_{k+1} & \dot{u}_{k+1} \end{vmatrix} - \beta \right)$$

which shows that  $\dot{J} = 0$  on solutions of (15).

The above illustrates that in the differential-difference case we deal with the two different types of integrals. Due to symmetries  $t \leftrightarrow x$  and  $k \leftrightarrow l$  in continuous and lattice cases, it is sufficient to indicate integrals with respect to one variable only.

### 3. Integrals for vanishing determinant

In what follows Sylvester's determinant identity [30] plays an important role. Let  $\mathcal{A}_n$  be the matrix obtained from  $\mathcal{A}$  by retaining its first  $n$  rows and columns and deleting the rest. According to the notation introduced earlier,  $\det \mathcal{A}_n = w_n$ . Further, we denote the minor of the entry in the  $p$ -th row and  $q$ -th column of matrix  $\mathcal{A}_n$  as  $m_{pq}$  and the minor obtained from  $\mathcal{A}_n$  by removing its  $p$ - and  $q$ -th rows as well as  $r$ - and  $s$ -th columns as  $m_{pqrs}$ . A particular case of Sylvester's identity (also known as Jacobi identity [31]), that we require, reads

$$m_{pqrs} w_n = m_{pr} m_{qs} - m_{ps} m_{qr}. \quad (19)$$

To indicate the order of minors  $m_{pq}$  and  $m_{pqrs}$  explicitly, we write  $m_{n;pq}$  and  $m_{n;pqrs}$  respectively. This notation means that the minors are obtained by deleting rows and columns from matrix  $\mathcal{A}_n$ . We will refer to particular instances of Sylvester's identity by indicating the list of indices  $(p, q, r, s)$ .

First we develop a procedure for construction of integrals of a vanishing determinant and then show (Section 5) that it can be easily extended to the non-vanishing case similarly to how it is done for the Liouville equation in the previous section.

Vanishing of determinant  $w_N$  implies a linear dependence of its columns or rows. Hence the last column and row can be decomposed correspondingly as

$$a_{iN} = \sum_{r=1}^{N-1} a_{ir} I_{N;r}, \quad a_{Ni} = \sum_{p=1}^{N-1} a_{pi} J_{N;p}. \quad (20)$$

Clearly the coefficients  $I_{N;r}$  do not depend on the variable that labels different rows of the determinant. Functions  $I_{N;r}$  are therefore the  $x$ -integrals in the continuous case and  $k$ -integrals in the semi-discrete and lattice cases. Likewise the coefficients  $J_{N;p}$  are the  $t$ -integrals in the continuous and semi-discrete cases and  $l$ -integrals in the lattice case. Setting  $i = 1, \dots, N-1$  in (20) and solving for  $I_{N;r}$  and  $J_{N;p}$  using Cramer's rule we get explicit expressions for the integrals of (1):

$$I_{N;r} = \frac{m_{Nr}}{w_{N-1}}, \quad J_{N;p} = \frac{m_{pN}}{w_{N-1}}, \quad p, r = 1, \dots, N-1. \quad (21)$$

For consistency of formulas that follow we also define

$$I_{N;N} = J_{N;N} = 1, \quad I_{N;0} = J_{N;0} = 0. \quad (22)$$

**Remark.** It is easy to see that homogeneous polynomials  $m_{Nr}$  are linearly independent and of the same order  $2(N-1)$ . On the other hand, each of these polynomials depends on a unique set of variables. This implies functional independence of integrals  $I_{N;r}$ . Apparently the same argument applies to integrals  $J_{N;p}$ .

Below we show that functions  $I_{n;r}$  and  $J_{n;p}$  satisfy some remarkable identities which imply representations of form (8), (10) and (11).

**Lemma 1.** *The functions  $I_{n;r}$  and  $J_{n;p}$  satisfy the following identities.*

*Continuous case:*

$$\partial_x I_{n;r} = \frac{w_{n-2}w_n}{w_{n-1}^2} I_{n-1;r}, \quad \partial_t J_{n;p} = \frac{w_{n-2}w_n}{w_{n-1}^2} J_{n-1;p}. \quad (23)$$

*Semi-discrete case:*

$$\Delta_k I_{n;r} = \frac{w_n}{w_{n-1}} S_k \left( \frac{w_{n-2}}{w_{n-1}} I_{n-1;r} \right), \quad \frac{dJ_{n;p}}{dt} = \frac{w_{n-2}w_n}{w_{n-1}^2} J_{n-1;p}. \quad (24)$$

*Lattice case:*

$$\Delta_k I_{n;r} = \frac{w_n}{w_{n-1}} S_k \left( \frac{w_{n-2}}{w_{n-1}} I_{n-1;r} \right), \quad \Delta_l J_{n;p} = \frac{w_n}{w_{n-1}} S_l \left( \frac{w_{n-2}}{w_{n-1}} J_{n-1;p} \right). \quad (25)$$

**Proof.** The idea of the proof is to match the derivative (difference) of  $I_{n;r}$  and  $J_{n;p}$  with the right-hand-side of Sylvester's identity for some  $(p, q, r, s)$ . We consider the semi-discrete case in detail as it is illustrative of both the continuous and lattice cases.

Note that the difference

$$\Delta_k I_{n;r} = S_k \left( \frac{m_{nr}}{w_{n-1}} \right) - \frac{m_{nr}}{w_{n-1}}$$

can be written as

$$\Delta_k I_{n;r} = \frac{1}{w_{n-1} S_k(w_{n-1})} (S_k(m_{nr})w_{n-1} - S_k(w_{n-1})m_{nr}). \quad (26)$$

Taking into account the relations

$$S_k(m_{nr}) = m_{1r}, \quad w_{n-1} = m_{nn}, \quad S_k(w_{n-1}) = m_{1n}, \quad m_{n;1nrn} = m_{n-1;1r},$$

and using Sylvester's identity with  $(1, n, r, n)$  we re-write the expression in brackets as

$$\begin{aligned} S_k(m_{nr})w_{n-1} - S_k(w_{n-1})m_{nr} &= m_{1r}m_{nn} - m_{1n}m_{nr} \\ &= m_{n-1;1r}w_n. \end{aligned}$$

After multiplying and dividing the right hand side of (26) by  $S_k(w_{n-2})$  and taking into account that  $m_{n-1;1r} = S_k(m_{n-1;n-1,r})$  we obtain formula (24)<sub>1</sub>.

Further, in the expression for the derivative

$$\frac{dJ_{n;p}}{dt} = \frac{1}{w_{n-1}^2} (\dot{m}_{pn}w_{n-1} - m_{pn}\dot{w}_{n-1})$$

one can easily recognise that

$$\dot{m}_{pn} = m_{p,n-1}, \quad w_{n-1} = m_{nn}, \quad \dot{w}_{n-1} = m_{n,n-1}, \quad m_{n;pn,n-1,n} = m_{n-1;p,n-1}.$$

Substituting the latter in the former, and taking into account Sylvester's identity with  $(p, n, n-1, n)$ , we obtain

$$\frac{dJ_{n;p}}{dt} = \frac{w_n m_{n-1;p,n-1}}{w_{n-1}^2}$$

and hence  $(24)_2$ .

The pairs of formulas  $(23)_2$ ,  $(24)_2$  and  $(24)_1$ ,  $(25)_1$  are identical, so are their proofs.

## 4. Equations for leading principal minors

### 4.1. Derivation of equations

In this section we derive the equations, or more correctly, the chains of equations satisfied by the leading principal minors of matrix  $\mathcal{A}_n$ . In the continuous case we get a version of the 2D Toda equation given by (2). It is natural to expect that in semi-discrete and lattice cases we get some analogs of the 2D Toda equation. The derivation relies on identities (23)-(25) for functions  $I$  and  $J$ .

**Theorem 1.** *Quantities  $w_n$  satisfy the following relations.*

*Continuous case:*

$$\partial_{tx}^2 \ln w_n = \frac{w_{n-1} w_{n+1}}{w_n^2}. \quad (27)$$

*Semi-discrete case:*

$$\Delta_k \frac{d}{dt} \ln w_n = \frac{w_{n+1}}{w_n} S_k \left( \frac{w_{n-1}}{w_n} \right). \quad (28)$$

*Lattice case:*

$$\Delta_l \frac{w_n}{S_k(w_n)} = -\frac{w_{n+1}}{S_k(w_n)} S_{kl}^2 \left( \frac{w_{n-1}}{w_n} \right). \quad (29)$$

**Proof.** The formula in the continuous case can be derived from  $(23)_2$  if we substitute  $p = n-1$  and notice that

$$J_{n;n-1} = \partial_x \ln w_{n-1}, \quad J_{n-1,n-1} = 1.$$

This results in down-shifted formula (27).

Similarly, in the semi-discrete case we set  $r = n-1$  in  $(24)_1$  to obtain

$$\Delta_k I_{n;n-1} = \frac{w_n}{w_{n-1}} S_k \left( \frac{w_{n-2}}{w_{n-1}} \right).$$

This relation combined with

$$I_{n;n-1} = \frac{d}{dt} \ln w_{n-1}$$

gives formula (28) down-shifted with respect to  $n$ .



Equation (29) is in fact equivalent to Sylvester's identity for the corner entries of matrix  $\mathcal{A}_n$ . Indeed, setting  $(p, q, r, s) = (1, n, 1, n)$  in (19) we obtain

$$w_n m_{1n1n} = m_{11} m_{nn} - m_{1n} m_{n1}$$

which can be re-written as

$$w_n S_{kl}^2(w_{n-2}) = S_{kl}^2(w_{n-1}) w_{n-1} - S_k(w_{n-1}) S_l(w_{n-1}). \quad (30)$$

Equation (30) is a different way to write (29). Indeed, if we divide (30) by  $S_{kl}^2(w_{n-1}) S_k(w_{n-1})$  and recognise that the right-hand side of the resulting expression is the difference

$$(1 - S_l) \frac{w_{n-1}}{S_k(w_{n-1})} = -\Delta_l \frac{w_{n-1}}{S_k(w_{n-1})}$$

we obtain (29) down-shifted with respect to  $n$ . This concludes the proof.

**Remark.** Equation (28) with boundary conditions (3) and (4) is equivalent (up to a simple transformation) to the system introduced in [18] as a discretisation of the classical 2D Toda system of type  $A_N$  while equation (30) is related to the Hirota equation [32].

**Theorem 3.** *The quantities  $I_{N;r}$ ,  $J_{N;p}$  are the integrals of equation (1) or equivalently of (27)-(29) with boundary conditions (3),(4), assuming  $\beta = 0$ .*

On substituting  $n = N$  and  $w_N = 0$  to (23)-(25) we note that the right-hand sides of these relations vanish which concludes the proof.

#### 4.2. Recurrent formulas for integrals

We have shown that the formulas for integrals obtained in Section 3 are suitable not only for scalar equation (1) but also for recurrences (27)-(29) with the boundary conditions (3), (4). However, the integrals contain mixed derivatives (shifts) that can be eliminated by means of the recurrences themselves. This would give a more natural expression for the integrals as they are usually given in terms of derivatives (shifts) with respect to one variable only. Therefore our objective in this section is to express the integrals  $I$  and  $J$  in terms of derivatives (shifts) of leading principal minors  $w_n$ .

**Theorem 4.** *Quantities  $I_{n;r}$  and  $J_{n;p}$  satisfy the following recurrent formulas.*

*Continuous case:*

$$J_{n;p} = J_{n-1;p-1} - \partial_x J_{n-1;p} + J_{n-1;p} \partial_x \ln \frac{w_{n-1}}{w_{n-2}}. \quad (31)$$

*The recurrent formula for  $x$ -integrals  $I_{n;r}$  is identical to (31) with  $x$  replaced by  $t$ .*

*Semi-discrete case:*

$$I_{n;r} = I_{n-1;r-1} - \dot{I}_{n-1,r} + I_{n-1;r} \frac{d}{dt} \ln \frac{w_{n-1}}{w_{n-2}}, \quad (32)$$

$$J_{n;p} = S_k(J_{n-1;p-1}) + \frac{w_{n-2}}{w_{n-1}} S_k\left(\frac{w_{n-1}}{w_{n-2}}\right) J_{n-1;p}. \quad (33)$$

*Lattice case:*

$$J_{n;p} = S_k(J_{n-1;p-1}) + \frac{w_{n-2}}{w_{n-1}} S_k\left(\frac{w_{n-1}}{w_{n-2}}\right) J_{n-1;p}. \quad (34)$$

The recurrent formula for  $k$ -integrals  $I_{n;r}$  is identical to (34) with  $k$  replaced by  $l$ .

The initial and boundary conditions for these formulae are given by (cf. (22))

$$J_{1,0} = 0, \quad J_{1,1} = 1, \quad J_{n,0} = 0, \quad J_{n,n} = 1. \quad (35)$$

*Proof. Continuous case.* We start with a relation that follows directly from the rule of differentiation of Wronskians:

$$m_{n-1;p-1,n-1} = \partial_x m_{n-1;p,n-1} - m_{p,n-1,n-1,n}. \quad (36)$$

Its last term can be re-written by means of Sylvester's identity the following way

$$\begin{aligned} w_n m_{p,n-1,n-1,n} &= m_{p,n-1} m_{n-1,n} - m_{pn} m_{n-1,n-1} \\ &= \partial_t m_{pn} \partial_x w_{n-1} - m_{pn} \partial_{tx}^2 w_{n-1}. \end{aligned}$$

Using (27) and (23)<sub>2</sub> written correspondingly in the form

$$\partial_{tx}^2 w_{n-1} = \frac{\partial_t w_{n-1} \partial_x w_{n-1}}{w_{n-1}} + \frac{w_{n-2} w_n}{w_{n-1}}, \quad \partial_t m_{pn} = \frac{m_{pn} \partial_t w_{n-1}}{w_{n-1}} + \frac{m_{n-1;p,n-1} w_n}{w_{n-1}}$$

we obtain

$$m_{p,n-1,n-1,n} = \frac{\partial_x w_{n-1} m_{n-1;p,n-1}}{w_{n-1}} - \frac{w_{n-2} m_{pn}}{w_{n-1}}. \quad (37)$$

Substituting this formula in (36) and dividing by  $w_{n-2}$ , we obtain the formula

$$\frac{m_{pn}}{w_{n-1}} = \frac{m_{n-1;p,n-1} \partial_x w_{n-1}}{w_{n-2} w_{n-1}} - \frac{\partial_x m_{n-1;p,n-1}}{w_{n-2}} + \frac{m_{n-1;p-1,n-1}}{w_{n-2}} \quad (38)$$

equivalent to (31).

*Semi-discrete case.* Here we aim to obtain recurrent formulas that yield derivative-free  $t$ -integrals and shift-free  $k$ -integrals. The constructions for  $k$ - and  $t$ -integrals appear to be slightly different, so we consider both in detail.

*Formula for  $k$ -integrals.* We start with the relation

$$m_{n-1;n-1,r} = \dot{m}_{n-1;n-1,r+1} - m_{n-1,n,r+1,n-1}$$

which is identical to (36). Its up-shifted version is given by

$$S_k(m_{n-1;n-1,r}) = S_k(\dot{m}_{n-1;n-1,r+1}) - m_{1,n,r+1,n-1}. \quad (39)$$

Consider Sylvester's identity with  $(1, n, r+1, n-1)$ :

$$w_n m_{1,n,r+1,n-1} = m_{1,r+1} m_{n,n-1} - m_{1,n-1} m_{n,r+1}. \quad (40)$$

The terms in the right-hand side of (40) can be interpreted as

$$m_{1,r+1} = S_k(m_{n,r+1}), \quad m_{n,n-1} = \dot{w}_{n-1}, \quad m_{1,n-1} = S_k(\dot{w}_{n-1}).$$

Substituting these expressions in (40) and also multiplying and dividing its right-hand side by  $w_{n-1}S_k(w_{n-1})$ , we obtain

$$w_n m_{1,n,r+1,n-1} = w_{n-1} S_k(w_{n-1}) \left( S_k(I_{n;r+1}) \frac{d}{dt} \ln w_{n-1} - I_{n;r+1} S_k\left(\frac{d}{dt} \ln w_{n-1}\right) \right). \quad (41)$$

Formula (41) can be simplified by means of the relations

$$\begin{aligned} I_{n;r+1} &= S_k(I_{n;r+1}) - \frac{w_n}{w_{n-1}} S_k\left(\frac{w_{n-2}}{w_{n-1}} I_{n-1;r+1}\right), \\ \frac{d}{dt} \ln w_{n-1} &= S_k\left(\frac{d}{dt} \ln w_{n-1}\right) - \frac{w_n}{w_{n-1}} S_k\left(\frac{w_{n-2}}{w_{n-1}}\right) \end{aligned}$$

which follow from (24)<sub>1</sub> and (28)<sub>1</sub> correspondingly. The formula then becomes

$$m_{1,n,r+1,n-1} = S_k\left(w_{n-2}\left(I_{n-1;r+1} \frac{d}{dt} \ln w_{n-1} - I_{n;r+1}\right)\right). \quad (42)$$

Substituting (42) back in (39) and down-shifting the obtained expression with respect to  $k$ , we get

$$m_{n-1;n-1,r} = \dot{m}_{n-1;n-1,r+1} - w_{n-2}\left(I_{n-1;r+1} \frac{d}{dt} \ln w_{n-1} - I_{n;r+1}\right),$$

or

$$I_{n-1;r} = \frac{\dot{m}_{n-1;n-1,r+1}}{w_{n-2}} - I_{n-1;r+1} \frac{d}{dt} \ln w_{n-1} + I_{n;r+1}. \quad (43)$$

On noting that

$$\frac{\dot{m}_{n-1;n-1,r+1}}{w_{n-2}} = \dot{I}_{n-1;r+1} + I_{n-1;r+1} \frac{d}{dt} \ln w_{n-2} \quad (44)$$

and down-shifting with respect to  $r$ , we obtain (32).

*Formula for  $t$ -integrals.* Our starting point is the relation

$$S_k(m_{n-1;p,n-1}) = m_{1,p+1,n-1,n}.$$

Sylvester's identity corresponding to indices  $(1, p+1, n-1, n)$  is given by

$$w_n m_{1,p+1,n-1,n} = m_{1,n-1} m_{p+1,n} - m_{1,n} m_{p+1,n-1}$$

or equivalently by

$$w_n m_{1,p+1,n-1,n} = m_{p+1,n} S_k(\dot{w}_{n-1}) - S_k(w_{n-1}) \dot{m}_{p+1,n}. \quad (45)$$

The mixed shift-derivative of  $w_{n-1}$  present in (45) can be eliminated by using (28) which, for convenience, we write as

$$S_k(\dot{w}_n) = \frac{1}{w_n} (S_k(w_n) \dot{w}_n + w_{n+1} S_k(w_{n-1})). \quad (46)$$

Further, the derivative  $\dot{m}_{p+1,n}$  can be eliminated by using (24)<sub>2</sub> written as

$$\dot{m}_{pn} = \frac{1}{w_{n-1}}(m_{pn}\dot{w}_{n-1} + m_{n-1;p,n-1}w_n). \quad (47)$$

Substituting (46) and (47) in (45) we obtain

$$m_{1,p+1,n-1,n} = \frac{1}{w_{n-1}}(m_{p+1,n}S_k(w_{n-2}) - m_{n-1;p+1,n-1}S_k(w_{n-1})). \quad (48)$$

On down-shifting with respect to  $p$ , dividing by  $S_k(w_{n-2})$  and re-arranging terms, relation (48) becomes (33).

*Lattice case.* Derivation of (34) is similar to the one for (33) in the semi-discrete case. We start with the obvious relation

$$S_{kl}^2(m_{n-1;p,n-1}) = m_{1,p+1,1,n}. \quad (49)$$

The right hand side is involved in the decomposition of  $w_n$  by means of Sylvester's identity:

$$\begin{aligned} w_n m_{1,p+1,1,n} &= m_{11}m_{p+1,n} - m_{1n}m_{p+1,1} \\ &= S_{kl}^2(w_{n-1})m_{p+1,n} - S_k(w_{n-1})S_l(m_{p+1,n}) \\ &= w_{n-1}S_{kl}^2(w_{n-1})J_{n;p+1} - S_k(w_{n-1})S_l(w_{n-1}J_{n;p+1}). \end{aligned}$$

Next, we can eliminate  $J_{n;p+1}$  from the first term on the right using formula (25)<sub>2</sub>. This yields the formula

$$\begin{aligned} w_n m_{1,p+1,1,n} &= S_l(J_{n;p+1})(w_{n-1}S_{kl}^2(w_{n-1}) - S_k(w_{n-1})S_l(w_{n-1})) \\ &\quad - w_n S_{kl}^2(w_{n-1})S_l\left(\frac{w_{n-2}}{w_{n-1}}J_{n-1;p+1}\right) \end{aligned}$$

which, in turn, can be simplified by using (30). The result is then given by

$$m_{1,p+1,1,n} = S_l\left(J_{n;p+1}S_k(w_{n-2}) - S_k(w_{n-1})\frac{w_{n-2}}{w_{n-1}}J_{n-1;p+1}\right).$$

Making the substitution given by the latter formula into (49) and down-shifting the resulting expression with respect to  $l$ , we obtain the formula

$$S_k(m_{n-1;p,n-1}) = J_{n;p+1}S_k(w_{n-2}) - S_k(w_{n-1})\frac{w_{n-2}}{w_{n-1}}J_{n-1;p+1}$$

which can be easily recognised as equivalent to (34).

**Remarks.** Obviously the following pairs of recurrent formulas are identical: (31) and (32), (33) and (34), where in the former one has to interchange  $x$  and  $t$ . This implies that the corresponding integrals are also identical.

There is an additional interpretation of the recurrent formulas (31)-(34): when they are combined with the corresponding relations from (23)-(25), the compatibility condition yields one of equations (27)-(29). Hence the combination gives a linear representation (Lax pair) for the corresponding equation.

### 4.3. Closed-form expressions

Recurrent formulas (31)-(34), due to conditions (35), can be summed to give closed-form expressions of functions  $I$  and  $J$  and hence the integrals of (27)-(29). It is sufficient to present them only for the continuous and lattice cases since the set of recurrent formulas in the semi-discrete case is the combination of the former two.

**Continuous case.** It is convenient to re-write formula (31) in the form

$$J_{p+i;p} = J_{p+i-1;p-1} - \frac{w_{p+i-1}}{w_{p+i-2}} \partial_x \left( \frac{w_{p+i-2}}{w_{p+i-1}} J_{p+i-1;p} \right)$$

for  $i \geq 1$ . For  $p = 1$  this formula yields

$$J_{i+1;1} = -\frac{w_i}{w_{i-1}} \partial_x \left( \frac{w_{i-1}}{w_i} J_{i;1} \right).$$

Further, setting  $p = 2, 3, \dots$  we obtain

$$J_{p+i;p} = \sum_{j=1}^p D_{i+j} (J_{j+i-1;j}), \quad (50)$$

where differential operator  $D_{i+j}$  has the form

$$D_{i+j} = -\frac{w_{i+j-1}}{w_{i+j-2}} \partial_x \frac{w_{i+j-2}}{w_{i+j-1}}.$$

Noting that (50) is a recurrence with respect to index  $i$  we make the self-substitution to express  $J_{p+i;p}$  in terms of  $J_{j+1;j} = \partial_x \ln w_j$  for some  $j$ . This yields the closed-form expression

$$J_{p+i;p} = \sum_{j_1=1}^p D_{i+j_1} \sum_{j_2=1}^{j_1} D_{i-1+j_2} \cdots \sum_{j_{i-1}=1}^{j_{i-2}} D_{2+j_{i-1}} J_{j_{i-1}+1,j_{i-1}}$$

**Discrete case.** As in the continuous case we start off by substituting  $n = i+1, p = 1$  in (34) to get

$$J_{i+1;1} = \frac{w_{i-1}}{w_i} S_k \left( \frac{w_i}{w_{i-1}} \right) J_{i;1}.$$

Continuing with  $n = p+i$  where  $p = 2, 3, \dots$  we observe that

$$J_{p+i;p} = \sum_{j=1}^p S_k^{p-j} \left( \frac{w_{i+j-2}}{w_{i+j-1}} S_k \left( \frac{w_{i+j-1}}{w_{i+j-2}} \right) J_{j+i-1;j} \right).$$

Again, this is a recurrence with respect to index  $i$ . The difference between continuous and discrete case is that now the expression for  $J_{j+1;j}$  is still quite complicated. This forces us to run the recurrence one step further down where we have  $J_{j;j} = 1$ . Thus the closed-form expression for  $J$  is

$$J_{p+i;p} = \sum_{j_1=1}^p S_k^{p-j_1} D_{i+j_1} \sum_{j_2=1}^{j_1} S_k^{j_1-j_2} D_{i-1+j_2} \cdots \sum_{j_i=1}^{j_{i-1}} S_k^{j_{i-1}-j_i} D_{1+j_i}$$

where

$$D_{i+j} = \frac{w_{i+j-2}}{w_{i+j-1}} S_k \left( \frac{w_{i+j-1}}{w_{i+j-2}} \right).$$

### 5. Non-vanishing case ( $\beta \neq 0$ )

Here we show that the integrals can be expressed in terms of functions  $I$  and  $J$  introduced before. We employ the approach used to derive integrals of the Liouville equation in Section 2. Namely, instead of the original equation we consider its differentiated (differenced) versions which have the form of a vanishing determinant.

Consider the new quantities  $\varphi_{n;r}$ ,  $\vartheta_{n;p}$  given by the following expressions.

Continuous case:

$$\varphi_{n;r} = I_{n;r} \partial_t \ln w_n - I_{n+1;r}, \quad \vartheta_{n;p} = J_{n;p} \partial_x \ln w_n - J_{n+1;p}. \quad (51)$$

Semi-discrete case:

$$\varphi_{n;r} = I_{n;r} \frac{d}{dt} \ln w_n - I_{n+1;r}, \quad \vartheta_{n;p} = S_k(J_{n;p}) + \frac{w_{n-1}}{S_k(w_{n-1})} J_{n;p+1}. \quad (52)$$

Lattice case:

$$\varphi_{n;r} = S_l(I_{n;r}) + \frac{w_{n-1}}{S_l(w_{n-1})} I_{n;r+1}, \quad \vartheta_{n;p} = S_k(J_{n;p}) + \frac{w_{n-1}}{S_k(w_{n-1})} J_{n;p+1}. \quad (53)$$

**Lemma 2.** *Quantities  $\varphi_{n;r}$ ,  $\vartheta_{n;p}$  satisfy the following relations.*

*Continuous case:*

$$\partial_x \varphi_{n;r} = I_{n-1;r} \frac{w_{n-2} \partial_t w_n}{w_{n-1}^2}, \quad \partial_t \vartheta_{n;p} = J_{n-1;p} \frac{w_{n-2} \partial_x w_n}{w_{n-1}^2}. \quad (54)$$

*Semi-discrete case:*

$$\Delta_k \varphi_{n;r} = S_k \left( \frac{w_{n-2}}{w_{n-1}} I_{n-1;r} \right) \frac{\dot{w}_n}{w_{n-1}}, \quad \dot{\vartheta}_{n;r} = S_k \left( \frac{w_{n-2}}{w_{n-1}^2} J_{n-1;p} \right) \Delta_k w_n. \quad (55)$$

*Lattice case:*

$$\Delta_k \varphi_{n;r} = S_{kl}^2 \left( \frac{w_{n-2}}{w_{n-1}} I_{n-1;r} \right) \frac{\Delta_l w_n}{S_l(w_{n-1})}, \quad \Delta_l \vartheta_{n;p} = S_{kl}^2 \left( \frac{w_{n-2}}{w_{n-1}} J_{n-1;p} \right) \frac{\Delta_k w_n}{S_k(w_{n-1})}. \quad (56)$$

The proof of these relations is quite straightforward. In order to prove, say (54)<sub>2</sub>, we differentiate (51)<sub>2</sub> with respect to  $t$  and use formulas (23) and (27) to eliminate the  $t$ -derivatives of  $J$  and the mixed derivative of  $w_n$ .

Proving (55) is slightly more involved: differentiating the expression for  $\vartheta_{n;r}$  and utilising (24) and (28), we get

$$\dot{\vartheta}_{n;r} = S_k \left( \frac{w_{n-2} w_n}{w_{n-1}^2} J_{n-1;p} \right) + \frac{w_{n-2} w_n}{w_{n-1} S_k(w_{n-1})} J_{n-1;p+1} - \frac{w_n S_k(w_{n-2})}{S_k(w_{n-1}^2)} J_{n;p+1}.$$

On using formula (33) up-shifted with respect to  $p$  to eliminate  $J_{n;p+1}$ , the previous formula simplifies to (55)<sub>2</sub>. Further, differencing the expression for  $\varphi_{n;r}$  and making use of formulas (24) and (28) we obtain (55)<sub>1</sub>.

Following the same strategy in the lattice case, we difference (53)<sub>2</sub> and use (25), (29) to obtain

$$\begin{aligned} \Delta_l \vartheta_{n;p} = S_k \left( \frac{w_n}{w_{n-1}} \right) S_{kl}^2 \left( \frac{w_{n-2}}{w_{n-1}} J_{n-1;p} \right) + \frac{w_n}{S_k(w_{n-1})} S_l \left( \frac{w_{n-2}}{w_{n-1}} J_{n-1;p+1} \right) \\ - \frac{w_n}{S_k(w_{n-1})} S_{kl}^2 \left( \frac{w_{n-2}}{w_{n-1}} \right) S_l(J_{n;p+1}). \end{aligned}$$

On eliminating  $J_{n;p+1}$  by means of (34) up-shifted with respect to  $p$ , we get (56)<sub>2</sub>. This concludes the proof.

**Theorem 6.** *The quantities  $\varphi_{N;r}$ ,  $\vartheta_{N;p}$  are the integrals of equation (1) or equivalently of (27)-(29) with boundary conditions (3),(4), assuming  $\beta \neq 0$ .*

**Proof.** Substituting  $n = N$  to (54)-(56) and noting that (4) implies that the derivatives (differences) of  $w_N$  vanish hence the right hand sides of (54)-(56) vanish as well. This concludes the proof.

## 6. General solutions

In this section we carry out calculations similar to the ones done for the continuous case in [33] which is also included here for the completeness of exposition.

The simplest equation (4) corresponds to  $N = 2$  and  $\beta = 0$ :

$$\begin{vmatrix} u & u_t \\ u_x & u_{tx} \end{vmatrix} = 0 \quad (57)$$

and is related to the D'Alembert equation  $v_{tx} = 0$  via the substitution  $u = \exp(v)$ . The general solution of (57) is therefore  $u = X_1(x)T_1(t)$ , where  $X_1$  and  $T_1$  are arbitrary functions. The form of equations and the solution in the simplest case suggest trying the ansatz

$$u(t, x) = \sum_{i=1}^{N-1} X_i(x)T_i(t) \quad (58)$$

for arbitrary  $N$ . Note that this ansatz contains the same number of arbitrary functions as the order of the equation hence it is a good candidate for the general solution.

After substituting (58) into matrix  $\mathcal{A}$  we can represent it as the product of two rectangular matrices:

$$\begin{bmatrix} u & u_t & \dots & u_{t^{N-1}} \\ u_x & u_{tx} & \dots & u_{t^{N-1}x} \\ \vdots & \vdots & \ddots & \vdots \\ u_{x^{N-1}} & u_{tx^{N-1}} & \dots & u_{t^{N-1}x^{N-1}} \end{bmatrix} = \begin{bmatrix} X_1 & X_2 & \dots & X_{N-1} \\ X'_1 & X'_2 & \dots & X'_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ X_1^{(N-1)} & X_2^{(N-1)} & \dots & X_{N-1}^{(N-1)} \end{bmatrix} \times \begin{bmatrix} T_1 & T'_1 & \dots & T_1^{(N-1)} \\ T_2 & T'_2 & \dots & T_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ T_{N-1} & T'_{N-1} & \dots & T_{N-1}^{(N-1)} \end{bmatrix}.$$

According to the Cauchy-Binet formula the determinant of  $\mathcal{A}$  is zero and hence (58) is the general solution of (1) in the vanishing case.

The general solution for a slightly more involved case of  $\beta \neq 0$  can be derived from the solution of  $w_N = 0$ . Indeed, the latter equation is equivalent to  $\partial_{tx}^2 \ln w_{N-1} = 0$  (see formula (27) ) hence

$$w_{N-1} = X(x)T(t). \quad (59)$$

On the other hand, the result of substitution of (58) to  $w_{N-1}$  can be written as the product of Wronskians of functions  $X_i$  and  $T_i$ :

$$w_{n-1} \left( \sum_{i=1}^{N-1} X_i T_i \right) = W(X_1, \dots, X_{N-1})(x) W(T_1, \dots, T_{N-1})(t).$$

Taking into account the remarks above and using the easily verifiable fact that  $w_{N-1}(XTu) = X^{N-1}T^{N-1}w_{N-1}(u)$ , we can write the general solution of the equation  $w_N = \beta$  in the form

$$u(t, x) = \frac{\beta^{\frac{1}{N}} \sum_{i=1}^N X_i T_i}{\sqrt[N]{W(X_1, \dots, X_N)(x) W(T_1, \dots, T_N)(t)}}. \quad (60)$$

**Remark.** Note that this solution contains two “extra” arbitrary functions since the equation is of the order  $2N - 2$  while the solution is made of  $2N$  arbitrary functions. The number of functions can be reduced by re-parametrisation

$$\frac{T_i}{T_1} \rightarrow T_{i-1}, \quad \frac{X_i}{X_1} \rightarrow X_{i-1}, \quad i = 2, \dots, N.$$

Following exactly the same reasoning we find general solutions for  $\beta = 0$  in the semi-discrete and lattice cases. The general solutions are correspondingly

$$u(t, k) = \sum_{i=1}^{N-1} T_i(t) K_i(k), \quad u(k, l) = \sum_{i=1}^{N-1} K_i(k) L_i(l). \quad (61)$$

In order to get solutions for  $\beta \neq 0$  in the semi-discrete case we note that

$$w_{N-1}(Tu) = w_{N-1}(u)T^{N-1}$$

and

$$w_{N-1} \left( \sum_{i=1}^{N-1} T_i(t) K_i(k) \right) = W(T_1, \dots, T_{N-1})(t) C(K_1, \dots, K_{N-1})(k),$$

where  $C$  stands for the Casoratian of sequences  $K_1, \dots, K_{N-1}$ . Hence the general solution can be written in the form

$$u(t, k) = \frac{\sum_{i=1}^N T_i(t) K_i(k)}{\sqrt[N]{W(T_1, \dots, T_N)(t)}},$$



where sequences  $K_i$  satisfy the condition

$$C(K_1, \dots, K_N)(k) = \beta. \quad (62)$$

For example, we can choose  $K_1, \dots, K_N$  to be a linearly independent set of solutions of the linear difference equation

$$K(k+N) + c_1(k)K(k+N-1) + \dots + c_{N-1}(k)K(k+1) + (-1)^N K(k) = 0,$$

where  $c_1(k), \dots, c_{N-1}(k)$  are arbitrary sequences. It follows from Abel's lemma that the Casoratian of such a set is constant. We only need to ensure that sequences  $K_i$  are scaled such that condition (62) is satisfied.

Similarly in the lattice case we have

$$w_{N-1} \left( \sum_{i=1}^{N-1} K_i(k) L_i(l) \right) = C(L_1, \dots, L_{N-1})(l) C(K_1, \dots, K_{N-1})(k).$$

Hence the general solution can be represented in the form

$$u(t, k) = \sum_{i=1}^N K_i(k) L_i(l),$$

where sequences  $K_i$  satisfy (62) and  $L_i$  satisfy a similar condition

$$C(L_1, \dots, L_N)(l) = 1.$$

Note that general solutions of (27)-(29) are obtained by simply substitution of the solutions obtained above in minors  $w_n$ .

## Concluding remarks

In this paper we have studied the integrability of certain determinantal equations. We have proved their Darboux integrability by constructing recurrent and closed-form formulas for integrals and corresponding integrating factors. The general solutions of equations associated with vanishing determinant are given explicitly while in the non-vanishing case they are expressed in terms of solutions of linear ordinary equations. A connection of the determinantal equations with analogues of the 2D Toda equation allows one to construct integrals and solutions for the latter as well.

Of course there are other systems that can be derived from the determinant. A few different ways to introduce them were already indicated in [5]. This line of research will be continued in a separate publication. Another problem yet to be considered is enumerating various reductions of the discrete analogues of the 2D Toda system. Since the underlying determinantal structure for both continuous and discrete equations is the same, then there should be no formal obstacles for carrying across the reductions from continuous case.

An important attribute of integrability which is not considered in this paper, is the structure of higher symmetries. For Darboux integrable equations the higher symmetries can be constructed from integrals. The only currently missing ingredient in this construction is the operators that map integrals to symmetries. These operators can be shown closely related (see e.g. Section 2) to the integrating factors constructed in this paper.

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